International Journal of Theoretical Physics, Vol. 9, No. 3 (1974), pp. 145–156

Infinite-Dimensional Representations of the Lorentz Group: The Complete Series

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Received: 6 November 1972

Abstract

The method used by Carmeli to obtain a new form for the principal series of representations of the group $SL(2, C)$ is further generalized to all completely irreducible (finite and infinite-dimensional) representations of that group. This is done, following Naimark, by extending the meaning of one of the parameters appearing in the formula for the operators of the principal series of representations. As a result a new form for the complete series of representations of the group $SL(2, C)$ is obtained which describes the transformation law of an infinite set of quantities under the group translation in a way which is very similar, but as a generalization, to the way spinors appear in the finite-dimensional case. The finite-dimensional representation is then discussed in details and the relation between the new set of quantities (which becomes finite in this case) and 2-component spinors is found explicitly.

1. *Introduction*

The generalization of 2-component spinors, which appear in describing the finite-dimensional representations of the group $SL(2, C)$ when realized in the space of polynomials, has recently been suggested by Carmeli (1970). He introduced an infinite set of quantities‡ associated with the principal series of representations of that group in a way which is very similar, but as a generalization, to the way spinors§ appear in describing the finite-dimensional representations. The transformation law of these quantities,

† Supported in part by the Colgate Research Council and the Sloan Foundations.

‡ Just as in the spinor case these quantities should be considered as functions of space-time when applied in physics.

§ Throughout this paper the term spinor is used to mean symmetrical spinor.

at the same time, defines a new form of the principal series of representations of $SL(2, C)$.† As is well known, spinors appear (up to factorial terms) as the coefficients of the polynomials of the space in which the representation is realized, whereas their transformation law provides another form of the representation (Naimark, 1964).

The method used for the principal series was subsequently extended by Carmeli & Malin (1971a, b) to the complementary series of representations, thus establishing new forms for all irreducible unitary representations (to within unitary equivalence) of the group $SL(2, C)$.

In this paper we extend Carmeli's result to all completely irreducible (finite- and infinite-dimensional) representations of the group $SL(2, C)$, thus obtaining a new form for the (infinite-dimensional) complete series of representations, on one hand, and recovering the (finite-dimensional) spinor representation written now in terms of our quantities instead of 2-component spinors, on the other hand.‡

In Section 2 we present our form for all the completely irreducible representations and compare it with that of Carmeli for the principal series. Section 3 is devoted to discussing the finite-dimensional representation. A detailed analysis is presented in order to establish a direct relation between our quantities for this case and the 2-component spinors. This relation is shown to be a simple linear transformation. In Appendix A we briefly review the spinor representation, whereas Appendix B is devoted to detailed calculations of certain functions appearing in the formula for the operators of the representation.

2. *The Complete Series of Representations*

The complete series describes all the infinite-dimensional irreducible representations, to within equivalence, of the group $SL(2, C)$. The meaning of equivalence here is such that the spaces of two equivalent representations need not be isometric, but it is the formulas which are essential for the representations and not the norm of the space. In fact, the formula of the representation of the complete series is the same as that of the principal series except for the meaning of one of the parameters whose value is now extended to the complex plane and, as a result, the representation ceases to be unitary§ in general (Gelfand & Naimark, 1947; Naimark, 1954).

† We recall that $SL(2, C)$ is, of course, the group of all 2×2 complex matrices with determinant unity, and it is the covering group of the restricted Lorentz group describing homogeneous Lorentz transformations which are orthochronous and proper. See, for example, R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).

‡ For the physical significance of nonunitary representations of the Lorentz group see A. O. Barut and S. Malin, *Review of Modern Physics*, 40, 632 (1968), Appendix B.

§ Every irreducible unitary representation of the group $SL(2, C)$ is unitarily equivalent to a representation of either the principal series or the complementary series of representations. These two series are, of course, included in the complete series as particular cases.

As for the principal and complementary series, the complete series may be realized in several ways according to the space of realization. For our purpose, just as for the principal series case, the particular realization of the complete series by means of the special unitary group SU_2, is employed.

We denote by $L_2^{2s}(SU_2)$ the set of all functions $\phi(u)$, where $u \in SU_2$, which are measurable and satisfying the conditions

$$\phi(\gamma u) = e^{is\psi}\phi(u) \tag{2.1}$$

$$\int |\phi(u)|^2\, du < \infty \tag{2.2}$$

where $\gamma \in SU_2$ is given by

$$\gamma = \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \tag{2.3}$$

Here $L_2^{2s}(SU_2)$ provides a Hilbert space (Naimark, 1964; Carmeli, 1969) where the scalar product is defined by†

$$(\phi_1, \phi_2) = \int \phi_1(u)\,\bar{\phi}_2(u)\, du \tag{2.4}$$

The complete series of representations, following the notation of Naimark, is then given by the formula (Naimark, 1964)

$$V_g\,\phi(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})}\phi(u\bar{g}) \tag{2.5}$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{2.6}$$

is an element of the group $SL(2, C)$, and $\alpha(g)$ is a function given by

$$\alpha(g) = g_{22}^{2s}|g_{22}|^{i\rho-2s-2} \tag{2.7}$$

Here ρ is a complex number and $2s$ is an integer, where $\rho^2 \neq -4(|s| + k)^2$, $k = 1, 2, 3, \ldots$ The expression $u\bar{g}$, appearing in the representation formula (2.5), denotes a matrix of the group SU_2 which is determined by means of u and $\bar{g}$ (up to an arbitrary phase factor) and whose explicit expression is given in Appendix B.

† The integrals in equations (2.2) and (2.4), and throughout this paper, are invariant integrals over the group SU_2 which satisfy the conditions

$$\int f(uu_1)\, du = \int f(u_1 u)\, du = \int f(u)\, du$$

for any $u_1 \in SU_2$, and

$$\int f(u^{-1})\, du = \int f(u)\, du$$

$$\int du = 1$$

Consider now all possible systems of numbers ϕ^j_m, where $m=-j, -j+1, \ldots, j$ and $j=|s|, |s|+1, |s|+2, \ldots$, with the condition

$$\sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} |\phi^j_m|^2 < \infty \tag{2.8}$$

The aggregate of all such systems ϕ^j_m forms a Hilbert space, which is denoted by l_2^{2s}, where the scalar product is defined by

$$\sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} \phi^j_m \bar{\psi}^j_m \tag{2.9}$$

for any two vectors ϕ^j_m and ψ^k_m of l_2^{2s}. With each vector $\phi^j_m \in l_2^{2s}$ we associate the function

$$\phi(u) = \sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} \phi^j_m T^j_m(u) \tag{2.10}$$

where $T^j_m(u)$ is the matrix element $T^j_{sm}(u)$ of irreducible representation of SU_2. Since (Naimark, 1964)

$$T^j_m(\gamma u) = e^{is\psi} T^j_m(u) \tag{2.11}$$

the function given by equation (2.10) belongs to the space $L_2^{2s}(SU_2)$. On the other hand every function in $L_2^{2s}(SU_2)$ can be written in the form (2.10) since $T^j_m(u)$ provide a complete orthogonal set (Carmeli, 1969).† The two spaces $L_2^{2s}(SU_2)$ and l_2^{2s} are, in fact, isometric where the transition from one space to the other can be made by means of the generalized Fourier transform

$$\phi^j_m = \int \phi(u) \bar{T}^j_m(u)\, du \tag{2.12}$$

Similarly to spinors which appear as coefficients in the polynomials of the space of representation (see Appendix A) we see that the numbers ϕ^j_m appear as coefficients in the expansion given by equation (2.10) of the functions $\phi(u)$ of the space $L_2^{2s}(SU_2)$. By means of the mapping (2.12) the operator V_g of the representation (2.5) may also be regarded as an operator in the space l_2^{2s} whose explicit expression is given below. This expression also defines another form of the complete series of representations.

Applying the operator V_g to the function $\phi(u)$ as given by equation (2.10) one obtains

$$V_g \phi(u) = \sum_j (2j+1) \sum_m \phi^j_m \frac{\alpha(ug)}{\alpha(u\bar{g})} T^j_m(u\bar{g}) \tag{2.13}$$

or

$$V_g \phi(u) = \sum_j (2j+1) \sum_m \phi^j_m \sum_{j'} (2j'+1) \sum_{m'} V^{jj'}_{mm'}(g; s, \rho) T^{j'}_{m'}(u) \tag{2.14}$$

† The functions $T^j_{mn}(u)$ satisfy the orthogonality condition

$$\int T^j_{mn}(u) \bar{T}^{j'}_{m'n'}(u)\, du = (2j+1)^{-1} \delta^{jj'} \delta_{mm'} \delta_{nn'}.$$

where

$$V_{mm'}^{jj'}(g;s,\rho) = \int \frac{\alpha(ug)}{\alpha(u\bar{g})} T_m^j(u\bar{g}) \bar{T}_{m'}^{j'}(u)\, du \tag{2.15}$$

Accordingly one obtains

$$V_g \phi(u) = \sum_j (2j+1) \sum_m \phi_m'^j T_m^j(u) \tag{2.16}$$

where, using equation (2.14), one obtains

$$\phi_{m'}'^{j'} = \sum_{j=|s|}^{\infty} (2j+1) \sum_{m=-j}^{j} V_{mm'}^{jj'}(g;s,\rho)\, \phi_m^j \tag{2.17}$$

Thus the operator V_g of the complete series of representations of $SL(2, C)$ in the space l_2^{2s} is the linear transformation determined by equation (2.17) describing the law of transformation of the quantities ϕ_m^j, where $j = |s|$, $|s|+1$, $|s|+2, \ldots$, and $m = -j, -j+1, \ldots, j$. The matrices $V_{mm'}^{jj'}(g;s,\rho)$ are functions of $g \in SL(2, C)$ and of ρ and s, where ρ is a complex number and $2s$ is an integer.

It will be noted that our formulas are identical to those of Carmeli (1970) for the principal series except for the meaning of the parameter ρ which is now complex. One can show that every completely irreducible representation of the group $SL(2, C)$ is defined by the pair of numbers (s,ρ) where $2s$ is some integer and ρ is some complex number; the pairs (s,ρ) and $(-s,-\rho)$ define the same completely irreducible representation. When $\rho^2 \neq -4(|s|+k)^2$, where $k = 1, 2, 3, \ldots$, then the representation is equivalent to the complete series;† when $\rho^2 = -4(|s|+k)^2$, $k = 1, 2, 3, \ldots$, the representation is equivalent to the finite-dimensional spinor representation of the group $SL(2, C)$. In the next section we find explicitly the relation between spinors and our quantities ϕ_m^j when the latter choice for ρ is made.

3. *Relation to Spinors*

We have seen in Section 2 that the representation formula (2.5), corresponding to all pairs (s,ρ), where $\rho^2 \neq -4(|s|+k)^2$ with $k = 1, 2, 3, \ldots$, describes the complete series of representations.

When $\rho^2 = -4(|s|+k)^2$, $k = 1, 2, 3, \ldots$, the representation (2.5) is not irreducible and one obtains the usual finite-dimensional spinor representation. This fact enables us to establish a direct relation between our quantities in the finite-dimensional representation and the 2-component spinors appearing in this case. We show below that this relation is a simple linear transformation.

† In view of the simplicity of formula (2.17) for the complete series, the question arises as to whether the complicated construction for the complementary series of Carmeli & Malin (1971a, b) was needed. The answer is, of course, that the representations in the complete series, which are equivalent to the complementary series, are not unitary whereas the representation given by Carmeli & Malin (1971a, b) is unitary.

To see that indeed when $\rho^2 = -4(|s| - k)^2$ the representation (2.5) is not irreducible we proceed as follows.

Suppose that $\rho = -2i(|s| - k)$ and denote by P_{MN} the set of all homogeneous polynomials in u_{21}, $\bar{u}_{21}$, u_{22} and $\bar{u}_{22}$:

$$p(u) = \sum_{\alpha,\beta,\gamma,\delta} a_{\alpha\beta\gamma\delta} u_{21}^{\alpha} \bar{u}_{21}^{\beta} u_{22}^{\gamma} \bar{u}_{22}^{\delta} \tag{3.1}$$

with the conditions

$$\alpha - \beta + \gamma - \delta = 2s \tag{3.2}$$

$$\alpha + \beta + \gamma + \delta = 2|s| + 2k - 2 \tag{3.3}$$

where $k = 1, 2, 3, \ldots$. One can easily see, using (3.2), that

$$p(\gamma u) = e^{is\psi} p(u) \tag{3.4}$$

where γ is given by (2.3). Therefore P_{MN} is a subspace of the Hilbert space $L_2^{2s}(SU_2)$. We show that P_{MN} is invariant with respect to the operator V_g of equation (2.5). To this end one writes†

$$g = u_1 \varepsilon u_2 \tag{3.5}$$

where $u_1, u_2 \in SU_2$ and ε is given by

$$\varepsilon = \begin{pmatrix} \varepsilon_{22}^{-1} & 0 \\ 0 & \varepsilon_{22} \end{pmatrix} \tag{3.6}$$

with ε_{22} a real number. Since $V_g = V_{u_1} V_\varepsilon V_{u_2}$, it is sufficient to show that P_{MN} is invariant under each of the operators V_{u_1}, V_ε and V_{u_2}. Now

$$V_{u_1} p(u) = \frac{\alpha(uu_1)}{\alpha(u\bar{u}_1)} p(u\bar{u}_1) \tag{3.7}$$

It is shown in Appendix B that $\alpha(uu_1)/\alpha(u\bar{u}_1)$ is equal to $\exp(2isA)$, where A is an arbitrary real number. Also, a direct calculation, using (B14), shows that

$$p(u\bar{u}_1) = \sum_{\alpha,\beta,\gamma,\delta} \exp[iA(-\alpha + \beta - \gamma + \delta)]\, a_{\alpha\beta\gamma\delta} (uu_1)_{21}^{\alpha} (\overline{uu_1})_{21}^{\beta} (uu_1)_{22}^{\gamma} (\overline{uu_1})_{22}^{\delta} \tag{3.8}$$

Hence, using the condition (3.2), one obtains

$$V_{u_1} p(u) = p(uu_1) \tag{3.9}$$

which shows that P_{MN} is invariant with respect to the operator V_{u_1} (and, of course, to V_{u_2}).

Similarly, P_{MN} is invariant with respect to V_ε, where

$$V_\varepsilon p(u) = \frac{\alpha(u\varepsilon)}{\alpha(u\bar{\varepsilon})} p(u\bar{\varepsilon}) \tag{3.10}$$

† Every $g \in SL(2, C)$ can be written in the form (3.5). See, for example, Naimark, 1964, p. 164.

In Appendix B it is shown that $\alpha(u\varepsilon)/\alpha(u\bar{\varepsilon})$ is equal to $\exp(2is\Lambda)|\lambda|^{i\rho-2}$, where $|\lambda|$ is given by (B17). Furthermore, one easily verifies that

$$p(u\bar{\varepsilon}) = \sum_{\alpha,\beta,\gamma,\delta} \exp[i\Lambda(-\alpha+\beta-\gamma+\delta)]|\lambda|^{-(\alpha+\beta+\gamma+\delta)}\varepsilon_{22}^{-\alpha-\beta+\gamma+\delta} \times a_{\alpha\beta\gamma\delta}u_{21}^{\alpha}\bar{u}_{21}^{\beta}u_{22}^{\gamma}\bar{u}_{22}^{\delta} \tag{3.11}$$

Using the conditions (3.2) and (3.3), and the fact that $\rho = -2i(|s|+k)$, one finds

$$V_{\varepsilon}p(u) = \sum_{\alpha,\beta,\gamma,\delta} \varepsilon_{22}^{-\alpha-\beta+\gamma+\delta} a_{\alpha\beta\gamma\delta}u_{21}^{\alpha}\bar{u}_{21}^{\beta}u_{22}^{\gamma}\bar{u}_{22}^{\delta} \tag{3.12}$$

This shows that $V_{\varepsilon}p(u)$ is a polynomial in the space P_{MN}. Hence P_{MN} is invariant with respect to the operator V_g, and therefore the representation (2.5) is not irreducible when $\rho = -2i(|s|-k)$, $k = 1, 2, 3, \ldots$.†

To see, in fact, that this is the spinor representation (see Appendix A) we put

$$M = s + \frac{i}{2}\rho - 1, \qquad N = -s + \frac{i}{2}\rho - 1 \tag{3.13}$$

Then, by (3.2), (3.3) and (3.13), one obtains

$$\gamma = M - \alpha, \qquad \delta = N - \beta \tag{3.14}$$

Accordingly, (3.1) can now be written as

$$p(u) = \sum_{\alpha=0}^{M}\sum_{\beta=0}^{N} a_{\alpha\beta}u_{21}^{\alpha}\bar{u}_{21}^{\beta}u_{22}^{M-\alpha}\bar{u}_{22}^{N-\beta} \tag{3.15}$$

Comparing (3.15) with (A16) we see that the quantity $a_{\alpha\beta}$ is just $\pi^{1/2}p_{rs}$. Hence $a_{\alpha\beta}$ is related to spinors, by (A6), by

$$a_{\alpha\beta} = \pi^{1/2}M!\,N!\,\phi_{A_1\ldots A_M\dot{X}_1\ldots\dot{X}_N} \tag{3.16}$$

with $A_1 + \cdots + A_M = \alpha$, $\dot{X}_1 + \cdots + \dot{X}_N = \beta$.

We are now in a position to find the connection between spinors and the quantities ϕ_m^J in the finite-dimensional case. Since $p(u) \in L_2^{2s}(SU_2)$, one can expand it into a generalized Fourier series,

$$p(u) = \sum_{J=|s|}^{\infty}(2j+1)\sum_{m=-J}^{J}\phi_m^J T_m^J(u) \tag{3.17}$$

where ϕ_m^J is related to $p(u)$ by

$$\phi_m^J = \int p(u)\bar{T}_m^J(u)\,du \tag{3.18}$$

Using the expression (3.15) for $p(u)$ in (3.18) one obtains

$$\phi_m^J = \sum_{\alpha=0}^{M}\sum_{\beta=0}^{N} \tilde{C}_{m\alpha\beta}^{JMN}a_{\alpha\beta} \tag{3.19}$$

† The representation (2.5) is not irreducible also when $\rho = 2i(|s|+k)$, $k = 1, 2, 3, \ldots$, since the pairs (s,ρ) and $(-s,-\rho)$ define the same completely irreducible representation,

where $\tilde{C}^{jMN}_{m\alpha\beta}$ are some numerical coefficients,

$$\tilde{C}^{jMN}_{m\alpha\beta} = \int \bar{T}^{j}_{m}(u)\, u^{\alpha}_{21}\, \bar{u}^{\beta}_{21}\, u^{M-\alpha}_{22}\, \bar{u}^{N-\beta}_{22}\, du \tag{3.20}$$

And in terms of 2-component spinors, by equation (3.16), one obtains

$$\phi^{j}_{m} = \sum_{\alpha=0}^{M} \sum_{\beta=0}^{N} C^{jMN}_{m\alpha\beta}\, \phi_{A_1 \ldots A_M \dot{X}_1 \ldots \dot{X}_N} \tag{3.21}$$

where

$$C^{jMN}_{m\alpha\beta} = \pi^{1/2} M!\, N!\, \tilde{C}^{jMN}_{m\alpha\beta} \tag{3.22}$$

Here $A_1 + \cdots + A_M = \alpha$, $\dot{X}_1 + \cdots + \dot{X}_N = \beta$.

Appendix A: Spinor Representation

For completeness we briefly outline in this Appendix the spinor representation of the group $SL(2, C)$. For more details the reader is referred to the classic book of Naimark (1964).

We denote by P_{mn} the aggregate of all polynomials $p(z, \bar{z})$ in the variable z and its complex conjugate $\bar{z}$ of degree not exceeding m in z and n in $\bar{z}$, where m and n are fixed non-negative integers determining the representation. The space P_{mn} is a linear vector space where the operation of addition and multiplication by a number are defined in the usual way for polynomials.

An element of the group $SL(2, C)$ will be denoted by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A1}$$

where a, b, c and d are complex numbers satisfying the condition

$$ad - bc = 1 \tag{A2}$$

Define the operator T_g in P_{mn} by

$$T_g p(z, \bar{z}) = (bz + d)^m (\bar{b}\bar{z} + \bar{d})^n p\left(\frac{az + c}{bz + d}, \frac{\bar{a}\bar{z} + \bar{c}}{\bar{b}\bar{z} + \bar{d}}\right) \tag{A3}$$

The correspondence $g \to T_g$ is a linear representation of the group $SL(2, C)$ as can be easily verified. This is the spinor representation of $SL(2, C)$ of dimension $(m + 1)(n + 1)$.

In order to relate this representation to the 2-component spinors, one realizes it in a somewhat different way.

One considers all systems of numbers $\phi_{A_1 \ldots A_m \dot{X}_1 \ldots \dot{X}_n}$, symmetrical in both the indices $A_1, \ldots, A_m$ and in $\dot{X}_1, \ldots, \dot{X}_n$ taking the values 0 and 1. The set of all such systems of numbers provides a linear space, denoted by S_{mn}, of dimension $(m + 1)(n + 1)$.

A one-to-one linear mapping between the spaces P_{mn} and S_{mn} can easily be established. To each system $\phi_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n} \in S_{mn}$ there corresponds the polynomial

$$p(z,\bar{z}) = \sum_{\substack{A_1,\ldots,A_m \\ \dot{X}_1,\ldots,\dot{X}_{12}}} \phi_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n} z^{A_1+\cdots+A_m}\bar{z}^{X_1+\cdots+X_n} \tag{A4}$$

of degree not exceeding m in z and n in $\bar{z}$, and therefore $p(z,\bar{z}) \in P_{mn}$. On the other hand every polynomial

$$p(z,\bar{z}) = \sum_{r,s} p_{rs} z^r \bar{z}^s \tag{A5}$$

in P_{mn} can be written in the form (A4) if one relates the ϕ's and p's by means of

$$\phi_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n} = \frac{1}{m!\,n!} p_{rs} \tag{A6}$$

with $A_1 + \cdots + A_m = r$, and $\dot{X}_1 + \cdots + \dot{X}_n = s$.

A second form of the spinor representation is then obtained if one applies the polynomials (A4) in equation (A3). One obtains

$$T_g p(z,\bar{z}) = \sum_{\substack{A_1,\ldots,A_m \\ \dot{X}_1,\ldots,\dot{X}_n}} \phi'_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n} z^{A_1+\cdots+A_m}\bar{z}^{X_1+\cdots+X_n} \tag{A7}$$

where we have used the notation

$$\phi'_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n} = \sum_{\substack{B_1,\ldots,B_m \\ \dot{Y},\ldots,\dot{Y}_n}} a_{A_1B_1}\cdots a_{A_mB_m}\bar{a}_{\dot{X}_1\dot{Y}_1}\cdots\bar{a}_{\dot{X}_n\dot{Y}_n}\phi_{B_1\ldots B_m\dot{Y}_1\ldots\dot{Y}_n} \tag{A8}$$

and where $a_{11} = a$, $a_{10} = b$, $a_{01} = c$, and $a_{00} = d$.

The linear transformation (A8) determines the operator T_g of the spinor representation in the space S_{mn}. The quantity $\phi_{A_1\ldots A_m\dot{X}_1\ldots\dot{X}_n}$ is a spinor having m undotted indices and n dotted indices.

Finally, let us denote $p(z,\bar{z})$ by $f(z)$, and let

$$\alpha(g) = g_{22}^m \bar{g}_{22}^n \tag{A9}$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

is an element of $SL(2, C)$. Equation (A3) can then be written in the form

$$T_g f(z) = \alpha(zg) f(z\bar{g}) \tag{A10}$$

Here z denotes a complex variable and also the matrix

$$z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \tag{A11}$$

and the matrix $z' = z\bar{g}$ amounts to a transformation in which the variable z goes over into the new variable

$$z' = g'_{21}/g'_{22} \tag{A12}$$

where the matrix $g' \in SL(2, \mathrm{C})$ is given by

$$g' = zg = \begin{pmatrix} g_{11} & g_{12} \\ g_{11}z + g_{21} & g_{12}z + g_{22} \end{pmatrix} \tag{A13}$$

So that the new variable z', according to (A12) and (A13), is given by†

$$z' = \frac{g_{11}z + g_{21}}{g_{12}z + g_{22}} \tag{A14}$$

If now we write‡

$$\phi(u) = \pi^{1/2}\alpha(u) f(z) \tag{A15}$$

where $u, z \in \tilde{z}$,§ and $z = u_{21}/u_{22}$, then

$$\phi(u) = \pi^{1/2} \sum_{r=0}^{m} \sum_{s=0}^{n} p_{rs} u_{21}^r u_{22}^{m-r} \bar{u}_{21}^s \bar{u}_{22}^{n-s} \tag{A16}$$

Hence $\phi(u)$ runs through all polynomials which are homogeneous in u_{21}, u_{22} of degree m and in $\bar{u}_{21}$, $\bar{u}_{22}$ of degree n, and p_{rs} are related to spinors by (A6). Let $\tilde{P}_{mn}$ denote the set of all such polynomials. Then $\tilde{P}_{mn}$ is the set of all polynomials homogeneous of degree $m+n$ in u_{21}, u_{22}, $\bar{u}_{21}$, $\bar{u}_{22}$, satisfying the condition

$$\phi(\gamma u) = e^{i(m-n)\psi/2} \phi(u) \tag{A17}$$

where γ is given by equation (2.3). The operators of the representation in the space $\tilde{P}_{mn}$ are then given by the formula

$$T_g \phi(u) = \frac{\alpha(ug)}{\alpha(u\bar{g})} \phi(u\bar{g}) \tag{A18}$$

where $\phi(u) \in \tilde{P}_{mn}$ and $u\bar{g}$ is a matrix of SU_2 whose explicit expression is given in Appendix B. Comparison of (A9) with (2.7) gives

$$m = \frac{i}{2}\rho + s - 1, \qquad n = \frac{i}{2}\rho - s - 1 \tag{A19}$$

Appendix B: The Matrix $u\bar{g}$

The expression $u\bar{g}$ appearing in the representation formula (2.5) and throughout this paper means, following Naimark, a unitary matrix which belongs to a right coset, $\tilde{u}\bar{g}$.‖ In this Appendix an explicit expression for $u\bar{g}$ is given, in terms of the two matrices u and $\bar{g}$.

† For more details, see Naimark, 1964, p. 142.

‡ The aggregate of functions $f(z)$ provides a Hilbert space in which one can also realize the complete series of representations. For more details, see Naimark, 1964, p. 170.

§ $\tilde{z}$ is the set of all matrices kg, where $g \in SL(2, \mathrm{C})$ is fixed and k varies through the entire group of matrices of the form given by (B3). For more details, see Naimark, 1964, p. 140.

‖ For details, see Naimark, 1964, Section 11, p. 154; and Carmeli & Malin, 1971a, footnote 7.

Let us denote the matrix $u\bar{g}$ by u'. Then u' can be written as

$$u' = \begin{pmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ -\bar{\beta}' & \bar{\alpha}' \end{pmatrix} \tag{B1}$$

with the condition

$$|\alpha'|^2 + |\beta'|^2 = 1 \tag{B2}$$

One can then show† that $ug = k.u\bar{g} = ku'$, where k is a matrix having the form

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \tag{B3}$$

where λ and μ are complex numbers, and $\lambda \neq 0$. If one denotes now ug by g', then one has $g' = ku'$, or explicitly

$$\begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ -\bar{\beta}' & \bar{\alpha}' \end{pmatrix} \tag{B4}$$

This gives

$$g'_{21} = -\lambda\bar{\beta}', \qquad g'_{22} = \lambda\bar{\alpha}' \tag{B5}$$

from which one obtains

$$\alpha' = \bar{g}'_{22}/\bar{\lambda}, \qquad \beta' = -\bar{g}'_{21}/\bar{\lambda} \tag{B6}$$

Furthermore, using the condition (B2), one obtains

$$|\lambda|^2 = |g'_{21}|^2 + |g'_{22}|^2 \tag{B7}$$

But $g' = ug$. Let us denote u by

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{B8}$$

and g by

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{B9}$$

then

$$\begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix} = \begin{pmatrix} \alpha g_{11} + \beta g_{21} & \alpha g_{12} + \beta g_{22} \\ -\bar{\beta} g_{11} + \bar{\alpha} g_{21} & -\bar{\beta} g_{12} + \bar{\alpha} g_{22} \end{pmatrix} \tag{B10}$$

If we write now $\lambda = |\lambda| \exp(i\Lambda)$, where Λ is some real number (phase), then one finally obtains for (B6) and (B7)

$$\begin{aligned} \alpha' &= (-\beta\bar{g}_{12} + \alpha\bar{g}_{22})|\lambda|^{-1} e^{i\Lambda} \\ \beta' &= (\beta\bar{g}_{11} - \alpha\bar{g}_{21})|\lambda|^{-1} e^{i\Lambda} \end{aligned} \tag{B11}$$

† See Naimark, 1964, p. 158.

and

$$|\lambda|^2 = |\beta\bar{g}_{11} - \alpha\bar{g}_{21}|^2 + |-\beta\bar{g}_{12} + \alpha\bar{g}_{22}|^2 \tag{B12}$$

Hence, $u\bar{g}$ is determined by means of u and $\bar{g}$ up to an arbitrary phase factor.

Equations (B11) and (B12) can now be used to calculate the ratio $\alpha(ug)/\alpha(u\bar{g})$, appearing in the representation formula (2.5), for two cases of particular interest:

(a) g is a unitary matrix, u_0,

$$u_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ -\bar{\beta}_0 & \bar{\alpha}_0 \end{pmatrix}; \qquad |\alpha_0|^2 + |\beta_0|^2 = 1 \tag{B13}$$

Then one obtains

$$\begin{aligned} \alpha' &= (-\beta\bar{\beta}_0 + \alpha\alpha_0)\, e^{iA} \\ \beta' &= (\beta\bar{\alpha}_0 + \alpha\beta_0)\, e^{iA} \\ |\lambda| &= 1 \end{aligned} \tag{B14}$$

and

$$\frac{\alpha(uu_0)}{\alpha(u\bar{u}_0)} = e^{2isA} \tag{B15}$$

(b) g is the matrix given by

$$\varepsilon = \begin{pmatrix} \varepsilon_{22}^{-1} & 0 \\ 0 & \varepsilon_{22} \end{pmatrix} \tag{B16}$$

where ε_{22} is real. One obtains

$$\begin{aligned} \alpha' &= \alpha\varepsilon_{22}|\lambda|^{-1}\, e^{iA} \\ \beta' &= \beta\varepsilon_{22}^{-1}|\lambda|^{-1}\, e^{iA} \\ |\lambda|^2 &= |\beta|^2\varepsilon_{22}^{-2} + |\alpha|^2\varepsilon_{22}^{2} \end{aligned} \tag{B17}$$

and

$$\frac{\alpha(u\varepsilon)}{\alpha(u\bar{\varepsilon})} = |\lambda|^{i\rho-2}\, e^{2isA} \tag{B18}$$

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